JOURNAL OF APPROXIMATION THEORY 80, 50–75 (1995)

# Multivariate Polynomial Interpolation to Traces on Manifolds

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*Communicated by Carl de Boor*

Received June 1, 1991; accepted in revised form March 23, 1993

A necessary and sufficient condition for polynomials defined on some (proper or improper) linear manifolds on $\mathbb{R}^k$ is given in order that they agree there with the traces of a polynomial on $\mathbb{R}^k$ (see H. A. Hakopian and A. A. Sahakian, *in* "Abstracts, International Workshop on Multivariate Interpolation and Approximation, Duisburg, 1989"). An inductive construction of this interpolating polynomial is obtained. The analogous interpolation on the sphere and with homogeneous polynomials is also presented, and some connections with other multivariate and finite element interpolations are explored. © 1995 Academic Press, Inc.

## 0. INTRODUCTION

We are interested in interpolating to polynomials given on $(k-s)$-dimensional linear manifolds obtainable as the intersection of hyperplanes from a given (multi)set $\mathscr{H}$ of hyperplanes in $\mathbb{R}^k$. We treat this problem in full generality, taking account of multiplicities by the corresponding matching of derivative information and also taking account of information at infinity in case of empty intersection of the corresponding hyperplanes.

We give necessary and sufficient conditions, in terms of consistency of the given data, for the existence and uniqueness of an interpolant from

$$\Pi_n = \Pi_n(\mathbb{R}^k),$$



0021-9045/95 $6.00

the space of polynomials on $\mathbb{R}^k$ of total degree $\leqslant n$, or from the space

$$\Pi_n^\infty = \Pi_n^\infty(\mathbb{R}^k)$$

of *homogeneous* polynomials of degree $n$, under the assumption that the data intended to prescribe some derivative of order $r$ on some linear manifold are indeed in the form of a polynomial of degree $\leqslant n-r$.

We start with some multivariate notation. Let $\alpha = (\alpha_1, ..., \alpha_k)$, $\beta = (\beta_1, ..., \beta_k) \in \mathbb{Z}_+^k$ be multi-indices, and let $x = (x_1, ..., x_k)$, $y = (y_1, ..., y_k) \in \mathbb{R}^k$ be $k$-vectors. Then we define

$$|\alpha| := \alpha_1 + \cdots + \alpha_k, \qquad \alpha! := \alpha_1! \cdots \alpha_k!, \qquad \alpha' := (\alpha_1, ..., \alpha_{k-1}),$$

$$x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \qquad x \cdot y := x_1 y_1 + \cdots + x_k y_k.$$

The notation $\alpha \leqslant \beta$ means that $\alpha_i \leqslant \beta_i$ for $i = 1, ..., k$.

For $P \in \Pi_n$ and $i = 1, ..., n$, we denote by $P^{[i]}$ the homogeneous component of $P$ of degree $i$; hence, we have

$$P = \sum_{i=0}^{n} P^{[i]}, \qquad P^{[i]} \in \Pi_i^\infty.$$

Let $D_y$ be the directional derivative along $y$:

$$D_y := \sum_{i=1}^{k} y_i \frac{\partial}{\partial x_i}.$$

More generally, for any sequence $Y = (y^1, ..., y^m)$ in $\mathbb{R}^k$ and corresponding multi-index $\alpha \in \mathbb{Z}_+^m$, we set

$$D_Y^\alpha := D_{y^1}^{\alpha_1} \cdots D_{y^m}^{\alpha_m}.$$

We also use

$$D_\infty^{[i]} P := P^{[n-i]}, \qquad \text{for} \quad P \in \Pi_n \text{ and } i = 0, ..., n.$$

Note that the definition of $D_\infty^{[i]}$ depends on the integer $n$ which specifies in which polynomial space we seek the interpolant.

Now let $\mathscr{H}$ be a given multiset of hyperplanes in $\mathbb{R}^k$. This means that each $H$ is of the form

$$H = \{x \in \mathbb{R}^k : n_H \cdot x = b_H\}$$

for some nonzero $k$-vector $n_H$ and some number $b_H$. Note that we do not exclude here the possibility of repetitions. For $H \in \mathscr{H}$, we denote by $H^0$ the $(k-1)$-dimensional subspace parallel to $H$, i.e.,

$$H^0 := \{x \in \mathbb{R}^k : n_H \cdot x = 0\}.$$

We denote by

$$\mathscr{L} = \mathscr{L}_{\mathscr{H}}$$

the collection of all linear manifolds obtainable as intersections of hyperplanes from $\mathscr{H}$ and include here even linear manifolds corresponding to empty intersections, i.e., having only improper points in a manner to be made clear in a moment. Thus, $\lambda \in \mathscr{L}$ if and only if

$$\lambda = \lambda_{\mathscr{M}} := \bigcap_{H \in \mathscr{M}} H$$

for some $\mathscr{M} \subseteq \mathscr{H}$. For $\lambda \in \mathscr{L}$, we denote by $\lambda^0$ its **improper part** and mean by that the linear subspace

$$\lambda^0 = \lambda^0_{\mathscr{M}} := \bigcap_{H \in \mathscr{M}} H^0$$

parallel to it. (The possibility that $\lambda = \lambda^0$ is, of course, not excluded.)

We call $\lambda$ **proper** in case $\lambda \neq \emptyset$, and call it **improper** otherwise. Further, the expression "manifold" will always mean "proper or improper linear manifold."

Consideration of improper manifolds requires some care when it comes to containment. For $\lambda_1, \lambda_2 \in \mathscr{L}$, we write

$$\lambda_1 \preccurlyeq \lambda_2$$

and say that $\lambda_1$ **is contained in** $\lambda_2$ exactly when

$$\lambda_1 \subseteq \lambda_2 \qquad \text{and} \qquad \lambda_1^0 \subseteq \lambda_2^0.$$

Consequently, we identify $\lambda_1$ and $\lambda_2$ exactly when

$$\lambda_1 \preccurlyeq \lambda_2 \qquad \text{and} \qquad \lambda_2 \preccurlyeq \lambda_1.$$

In particular, we identify two improper hyperplanes exactly when their improper parts coincide.

We classify manifolds by their dimension, defined as follows:

$$\dim \lambda := \begin{cases} \dim \lambda^0, & \text{if } \lambda \text{ is proper,} \\ \dim \lambda^0 - 1, & \text{if } \lambda \text{ is improper.} \end{cases}$$

Now we obtain the nice relation which also is a motivation for the above classification,

$$\dim(\lambda \cap H) = \begin{cases} \dim \lambda, & \text{if } \lambda \preccurlyeq H, \\ \dim \lambda - 1, & \text{otherwise,} \end{cases}$$

for any $\lambda \in \mathscr{L}$ and any $H \in \mathscr{H}$.

For a proper manifold $\lambda$, we denote by $\lambda_\infty$ the improper manifold for which $(\lambda_\infty)^0 = \lambda^0$. $\lambda_\infty$ is the unique improper manifold of dimension $\dim \lambda - 1$ contained in $\lambda$ (but need not be in $\mathscr{L}$).

We write

$$\mathscr{L}^s = \mathscr{L}^s_{\mathscr{H}} := \{\lambda \in \mathscr{L} : \dim \lambda = k - s\}$$

for the collection of all manifolds in $\mathscr{L}$ of *co*dimension $s$. In particular,

$$\mathscr{L}^0 = \{\mathbb{R}^k\}, \qquad \mathscr{L}^1 = \{H \in \mathscr{H}\}.$$

The multiplicity of $\lambda \in \mathscr{L}$ plays a central role. It is, by definition, the cardinality of the multiset

$$\mathscr{H}_\lambda := \{H \in \mathscr{H} : \lambda \preccurlyeq H\}.$$

Let $\lambda \in \mathscr{L}^s$. Then $\# \mathscr{H}_\lambda \geqslant s$, with $\# \mathscr{H}_\lambda = s$ referred to as the **simple** case. In any case, we intend to prescribe on each $\lambda \in \mathscr{L}^s$ all derivatives in $\mathbb{R}^k$ normal to $\lambda$ and of order $\leqslant (\# \mathscr{H}_\lambda - s)$.

Moreover, in the inductive proof we will deal with the situation when $\mathbb{R}^k$ is replaced by some manifold $\Lambda$ containing $\lambda$. For this purpose, we choose for

$$\lambda \preccurlyeq \Lambda, \qquad m := \dim \Lambda - \dim \lambda, \qquad m_0 := \dim \Lambda^0 - \dim \lambda^0,$$

some orthonormal coordinate system

$$\lambda^\perp(\Lambda) := (\lambda_1^\perp(\Lambda), ..., \lambda_{m_0}^\perp(\Lambda))$$

for the **orthogonal complement of** $\lambda^0$ **in** $\Lambda^0$, which we will also denote by

$$\lambda^\perp(\Lambda).$$

If $\alpha \in \mathbb{Z}_+^m$, then $D^\alpha_{\lambda^\perp(\Lambda)}$ is, so far, only defined when $m_0 = m$, i.e., when both $\lambda$ and $\Lambda$ are of the same type, i.e., both proper or both improper. In the contrary case, i.e., when $\lambda$ is improper and $\Lambda$ is proper, then $m_0 = m - 1$, and then we define

$$D^\alpha_{\lambda^\perp(\Lambda)} := D^{\alpha'}_{\lambda^\perp(\Lambda)} D^{[\alpha_m]}_\infty, \qquad \text{if} \quad m_0 = m - 1.$$

(Since $\lambda \preccurlyeq \Lambda$, the case $\lambda$ proper and $\Lambda$ improper cannot occur.)

In the following, *we will omit* in the above and other notation the manifold $\Lambda$ iff $\Lambda = \mathbb{R}^k$.

For proper $\lambda \in \mathscr{L}$, we denote by

$$\Pi_n(\lambda)$$

the space of all polynomials of degree $\leqslant n$ on $\lambda$ in the coordinates with respect to any particular coordinate system on $\lambda$. Correspondingly,

$$\Pi_n^\infty(\lambda)$$

is the subspace of $\Pi_n(\lambda)$ of all *homogeneous* polynomials of degree $n$ on $\lambda$. In order to fix the latter class of polynomials, we choose the projection

$$0_\lambda$$

of the origin $0 \in \mathbb{R}^k$ into $\lambda$ as the coordinate origin on $\lambda$.

For improper $\lambda$, we take

$$\Pi_n(\lambda) := \Pi_n^\infty(\lambda^0).$$

We also put $\Pi_n(\lambda) = \mathbb{R}$ in case $\lambda$ is a point.

For $P \in \Pi_n$, $P|_\lambda$ denotes the **trace** of $P$ on $\lambda$ in case $\lambda$ is proper. For an improper $\lambda$, we define

$$P|_\lambda := P^{[n]}|_{\lambda^0} = D_\infty^{[0]} P|_{\lambda^0}.$$

For proper $l \in \mathscr{L}$, we denote by

$$\Pi_n(\lambda, l) \qquad \text{and} \qquad \Pi_n^\infty(\lambda, l)$$

the respective classes of polynomials on $\lambda$ with coefficients in $\Pi_m(l)$ for some $m$. For a proper $\lambda \preccurlyeq \Lambda$, there corresponds in a natural way to any $P \in \Pi_n(\Lambda)$ an element of $\Pi_n(\lambda, \lambda^\perp(\Lambda))$, which we will denote by

$$P^\lambda. \tag{0.1}$$

## 1. Consistency of the Data

Suppose $\mathscr{L}^s \neq \varnothing$ for some fixed $s$, $1 \leqslant s \leqslant k$. Consider the following sequence of polynomials:

$$\mathscr{D}^s = \mathscr{D}^s_{\mathscr{H}} := (P_{\lambda,\alpha} \in \Pi_{n-|\alpha|}(\lambda) \colon \lambda \in \mathscr{L}^s, \alpha \in \mathbb{Z}^s_+, |\alpha| \leqslant \#\mathscr{H}_\lambda - s).$$

This sequence will serve as data on the manifolds in $\mathscr{L}^s$, which means the *interpolation problem* is to find a polynomial $P \in \Pi_n$ such that

$$(D^\alpha_{\lambda^\perp} P)|_\lambda = P_{\lambda,\alpha}, \qquad \forall (\lambda, \alpha) \in \operatorname{dom} \mathscr{D}^s, \tag{1.1}$$

where $\operatorname{dom} \mathscr{D}^s$ denotes the set of all index pairs $(\lambda, \alpha)$ in the definition of $\mathscr{D}^s$ and, to recall, $\lambda^\perp = \lambda^\perp(\mathbb{R}^k)$. In effect, we want to recover $P$ from its traces and the traces of some of its derivatives on all the manifolds in $\mathscr{L}^s$.

If $s=0$, there is nothing to do, since $\mathscr{D}^0_{\mathscr{H}}$ has just one term, corresponding to the unique pair, $(\mathbb{R}^k, 0)$, in its domain; hence, $P=P_{\mathbb{R}^k,0}$ is the solution. For $s>0$, we carry out this recovery one dimension at a time, obtaining from the given data $\mathscr{D}^s$ a corresponding polynomial sequence $\mathscr{D}^{s-1}_{\mathscr{H}}$, then a corresponding $\mathscr{D}^{s-2}_{\mathscr{H}}$, and so on, until we have constructed the corresponding $\mathscr{D}^0_{\mathscr{H}}$ and thereby the desired $P$. This means that it is sufficient to consider the problem of extending the given data from the manifolds in $\mathscr{L}^s$ to the manifolds in $\mathscr{L}^{s-1}$, thereby constructing $\mathscr{D}^{s-1}$ from $\mathscr{D}^s$. For this, we make use of the fact that, if there is a polynomial $P$ satisfying (1.1), then the polynomials in the corresponding sequence $\mathscr{D}^{s-1}$ satisfy

$$(D^{\beta}_{\Lambda^\perp} P)|_{\Lambda} = P_{\Lambda,\beta}, \qquad \forall(\Lambda,\beta)\in \operatorname{dom}\mathscr{D}^{s-1}. \tag{1.2}$$

At this point, we can easily obtain some information about the polynomials in $\mathscr{D}^{s-1}$. In particular, we can easily find the polynomials

$$P^{\beta}_{\lambda,\Lambda,j} := D^{j}_{\lambda^\perp(\Lambda)} P_{\Lambda,\beta}|_{\lambda}, \qquad j=0, ..., \#\mathscr{H}_\lambda - s - |\beta|, \tag{1.3}$$

for arbitrary proper manifolds $\lambda \preccurlyeq \Lambda$, $\lambda\in\mathscr{L}^s$ and $(\Lambda,\beta)\in\operatorname{dom}\mathscr{D}^{s-1}$. In fact,

$$P^{\beta}_{\lambda,\Lambda,j} = \sum_{\gamma\in\mathbb{Z}^s_+,\ |\gamma|=|\beta|+j} c_\gamma P_{\lambda,\gamma}, \tag{1.4}$$

where the coefficients $c_\gamma$ are to be found from the relation

$$D^{\beta}_{\Lambda^\perp} D^{j}_{\lambda^\perp(\Lambda)} = \sum_{\gamma\in\mathbb{Z}^s_+,\ |\gamma|=|\beta|+j} c_\gamma D^{\gamma}_{\lambda^\perp}.$$

If the above $\lambda$ is improper, which in this case means that $\lambda=\Lambda_\infty$, then we can determine the polynomials

$$P^{\beta}_{\Lambda_\infty,\Lambda,j} := D^{[j]}_{\infty} P_{\Lambda,\beta}, \qquad j=0, ..., \#\mathscr{H}_\lambda - s - |\beta|, \tag{1.5}$$

using the equation

$$P^{\beta}_{\Lambda_\infty,\Lambda,j} = \sum_{\gamma\in\mathbb{Z}^s_+,\ |\gamma|=|\beta|+j} \frac{(a_\Lambda)^{\gamma'-\beta}}{(\gamma'-\beta)!} P_{\Lambda_\infty,\gamma}(\cdot - a_\Lambda), \tag{1.6}$$

where

$$a_\Lambda := \overline{00_\Lambda};$$

hence, the vector $a_\Lambda$ lies in $\Lambda^\perp$.

Fix now any particular index $(\Lambda,\beta)\in\operatorname{dom}\mathscr{D}^{s-1}$ with a proper $\Lambda$, and consider the task of combining the information obtained about $P_{\Lambda,\beta}$ into an appropriate data sequence.

We associate with $(\Lambda, \beta)$ the following collection of manifolds:

$$\mathscr{L}^v(\Lambda) = \mathscr{L}^v(\Lambda, \mathscr{H}) := \{\lambda \in \mathscr{L}^v : \lambda \preccurlyeq \Lambda\}, \qquad v = s, s+1.$$

The collection $\mathscr{L}^s(\Lambda)$ can contain at most one improper element, viz., $\Lambda_\infty$. In any case, we set

$$\mu_\lambda := \#\mathscr{H}_\lambda - s + 1 - |\beta| + 1, \qquad \lambda \in \mathscr{L}^s(\Lambda),$$

and

$$\mu_\infty := \begin{cases} \mu_{\Lambda_\infty}, & \text{if } \Lambda_\infty \in \mathscr{L}^s(\Lambda), \\ 0, & \text{otherwise.} \end{cases}$$

The "multiplicity" $\mu_\lambda$ is related to the upper limit of $j$ in (1.3) or in (1.5).

Next we set

$$\bar{\mu}_\lambda := \#(\mathscr{H}_\lambda \backslash \mathscr{H}_\Lambda), \qquad m_\Lambda := \#\mathscr{H}_\Lambda - (s-1) - |\beta|,$$

$$\bar{\mu}(\Lambda) := \sum_{\lambda \in \mathscr{L}^s(\Lambda)} \bar{\mu}_\lambda, \qquad \mu(\Lambda) := \sum_{\lambda \in \mathscr{L}^s(\Lambda)} \mu_\lambda,$$

and, for $l \in \mathscr{L}^{s+1}$,

$$\mathscr{L}^s_l(\Lambda) := \{\lambda \in \mathscr{L}^s : l \preccurlyeq \lambda \preccurlyeq \Lambda\},$$

$$\bar{\mu}_l(\Lambda) := \sum_{\lambda \in \mathscr{L}^s_l(\Lambda)} \bar{\mu}_\lambda = \#(\mathscr{H}_l \backslash \mathscr{H}_\Lambda), \qquad \mu_l(\Lambda) := \sum_{\lambda \in \mathscr{L}^s_l(\Lambda)} \mu_\lambda.$$

Note that

$$\mu_\lambda = \bar{\mu}_\lambda + m_\Lambda, \qquad \lambda \in \mathscr{L}^s(\Lambda).$$

We now consider the data on manifolds in $\mathscr{L}^s(\Lambda)$, i.e., the polynomial sequence

$$\mathscr{D}^{s,\beta}(\Lambda) := (P_{\lambda,j} : \lambda \in \mathscr{L}^s(\Lambda), j = 0, ..., \mu_\lambda - 1).$$

where $P_{\lambda,j} := P^\beta_{\lambda,\Lambda,j}$, and let

$$n_\lambda := \lambda^\perp(\Lambda)$$

be a unit normal for $\lambda$ in $\Lambda$.

The *interpolation problem* here is to find $q \in \Pi_n(\Lambda)$ (which is supposed to be $P_{\Lambda,\beta}$, see (1.3)) such that

$$D^j_{n_\lambda} q|_\lambda = P_{\lambda,j}, \qquad \forall(\lambda, j) \in \operatorname{dom} \mathscr{D}^{s,\beta}(\Lambda). \tag{1.7}$$

This problem is much simpler than the problem (1.1). Indeed, the multi-index $\alpha$ and the system $\lambda^\perp$ of vectors in (1.1) are replaced here by the nonnegative integer $j$ and the single vector $n_\lambda$. In effect, we have here the case of codimension one since the codimension of the manifolds $\lambda \in \mathcal{L}^s(\Lambda)$ in $\Lambda$ is one.

Essentially, the method we use in this paper is to reduce the problem (1.1) to a number of problems of type (1.7), i.e., to problems of codimension one.

In order for the problem (1.7) (or (1.1)) to be solvable, the given data sequence must satisfy some **consistency conditions**, which we now discuss. Roughly speaking, if

$$l = \lambda_1 \cap \lambda_2$$

for some $\lambda_1, \lambda_2 \in \mathcal{L}^s(\Lambda)$, then the data sequence provides information about the behavior on $l$ in two different ways; hence, a solution can exist only if these two sources do not contradict each other.

For example, if $l$ is proper, then, of course, problem (1.7) cannot be solvable unless

$$P_{\lambda_1,0} = P_{\lambda_2,0} \qquad \text{on } l.$$

This is the essence of consistency condition (a) below.

The second consistency condition, (b), concerns the case of improper $l$ (i.e., the case when $\lambda_1$ and $\lambda_2$ are parallel) and guarantees, for instance, the relation

$$P^{[0]}_{\lambda_1,j} = P^{[0]}_{\lambda_2,j}(\cdot + \overline{0_{\lambda_1}0_{\lambda_2}}).$$

The third consistency condition, (c), arises only if $\Lambda_\infty \in \mathcal{L}^s(\Lambda)$ and is similar to condition (b). In order to present the consistency conditions in full generality, we set

$$u_\Lambda := (n+s+1-\#\mathcal{H})_+ \operatorname{signum}(m_\Lambda),$$

and, for $l \in \mathcal{L}^{s+1}$,

$$v_l(\Lambda) := \min\{w, \mu_l(\Lambda) - 2\},$$

where

$$w := \bar{\mu}_l(\Lambda) + m_\Lambda + u_\Lambda - 2 = \#\mathcal{H}_l - (s+1) - |\beta| + u_\Lambda.$$

DEFINITION 1.1. For $0 < s < k$, proper $\Lambda \in \mathcal{L}^{s-1}$, and $\beta \in \mathbb{Z}^{s-1}_+$ with $|\beta| \leqslant \#\mathcal{H}_\Lambda - s + 1$, we say that the derived sequence $\mathcal{D}^{s,\beta}(\Lambda)$ is **consistent at** $\Lambda$ if it satisfies the following three conditions:

(a) For every proper $l \in \mathcal{L}^{s+1}(\Lambda)$, there exist polynomials

$$Q^\beta_{l,v} \in \Pi^\infty_v(l^\perp(\Lambda), l), \qquad v = 0, ..., v_l(\Lambda),$$

satisfying the relation

$$D_{n_\lambda}^j Q_{l,\nu}^\beta(t) = \frac{\nu!}{(\nu-j)!} D_t^{\nu-j} P_{\lambda,j}|_l, \tag{1.8}$$

for $\lambda \in \mathscr{L}_l^s(\Lambda)$, $j = 0, ..., \min\{\nu, \mu_\lambda - 1\}$, and $t \in \lambda^0 \cap l^\perp(\Lambda)$.

(b) For every improper $l \in \mathscr{L}^{s+1}(\Lambda)$, there exist polynomials

$$Q_{l,\nu}^\beta \in \Pi_\nu(l^\perp(\Lambda), l^0), \qquad \nu = 0, ..., v_l(\Lambda),$$

satisfying the relation

$$D_{n_\lambda}^j Q_{l,\nu}^\beta(t) = (D_\infty^{[\nu-j]} P_{\lambda,j})(\cdot + a_\lambda) \tag{1.9}$$

for each proper $\lambda \in \mathscr{L}_l^s(\Lambda)$, $j = 0, ..., \min\{\nu, \mu_\lambda - 1\}$, and $t \in \lambda \cap l^\perp(\Lambda)$.

(c) If $\lambda = \Lambda_\infty \in \mathscr{L}^s$, then the polynomials $Q_{l,\nu}^\beta$ defined in (b) satisfy the conditions

$$D_{l^\perp(\Lambda)}^j Q_{l,\nu}^\beta(0) = (D_{l^\perp(\Lambda)}^j P_{\lambda,\nu})|_{l^0 + a_\Lambda} \tag{1.10}$$

for $0 \leqslant j \leqslant \nu < \mu_\lambda - 1$.

*Remark* 1.1.1. It is not hard to check that conditions (a), (b), (c) are necessary for interpolation problem (1.1) to be solvable. Indeed, if there exists a polynomial $P \in \Pi_n$ satisfying (1.1), then the polynomials $Q_{l,\nu}^\beta$ can be chosen as follows:

In case (a),

$$Q_{l,\nu}^\beta(t) = (D_t^\nu D_{\Lambda^\perp}^\beta P)|_l, \qquad t \in l^\perp(\Lambda);$$

to check (1.8), it is enough to use the relation

$$D_a F(x) = m D_a D_x^{m-1} f$$

with $F(x) := D_x^m f$.

In cases (b) and (c) (see (0.1)),

$$Q_{l,\nu}^\beta = D_\infty^{[\nu]} f^{l^0}(t),$$

where $f := D_{\Lambda^\perp}^\beta P|_\Lambda$.

*Remark* 1.1.2. It is evident that all the interpolating parameters on the right-hand sides of (1.8) and (1.9) (for fixed $\nu$) are uniquely determined by $\nu + 1$ Hermitian parameters from among them. This follows by univariate Hermite interpolation (on the line in case (b) and on the circle by homogeneous polynomials in case (a); see Theorem 4.2 below for $k = 1$).

This also makes clear that conditions (1.8) and (1.9) do not impose any restriction in the case $\nu > \mu_l(\Lambda) - 2$.

DEFINITION 1.2. We say that $\mathscr{D}^{s,\beta}(\Lambda)$ is **fully consistent** if condition (c) holds and conditions (a) and (b) hold for $\nu = 0, ..., \mu_l(\Lambda) - 2$, i.e., $\upsilon_l(\Lambda) \geqslant \mu_l(\Lambda) - 2$.

Let us now consider the case $s = k$ (pointwise interpolation).

DEFINITION 1.3. For $\Lambda \in \mathscr{L}^{k-1}$ a proper line with directional unit vector $d_\Lambda$, and $\beta \in \mathbb{Z}_+^{k-1}$ with $|\beta| \leqslant \mathscr{H}_\Lambda - k + 1$, we will say that the derived sequence $\mathscr{D}^{k,\beta}(\Lambda)$ (of numbers) is **consistent at** $\Lambda$ if either $\mu(\Lambda) \leqslant n + 1 - |\beta|$ or

(d) there exists a polynomial $\tilde{P}_{\Lambda,\beta} \in \Pi_{n - \mu_\infty - |\beta|}(\Lambda)$ satisfying

$$D^j_{d_\Lambda} \tilde{P}_{\Lambda,\beta}(\lambda) = P_{\lambda,0} - D^j_{d_\Lambda} \bar{P}_{\Lambda,\beta} \tag{1.11}$$

for $j = 0, ..., \mu_\lambda - 1$ and every proper $\lambda \in \mathscr{L}^k(\Lambda)$, where

$$\bar{P}_{\Lambda,\beta} := \sum_{\nu=0}^{\mu_\infty - 1} P^\beta_{\Lambda_\infty,\nu}. \tag{1.12}$$

Note that in this case $P_{\lambda,0} \in \mathbb{R}$ and the number of conditions in (1.11) equals $\mu(\Lambda) - \mu_\infty$. Hence for $\mu(\Lambda) \leqslant n + 1 - |\beta|$, by Hermite interpolation, there exists a polynomial $\tilde{P}_{\Lambda,\beta}$ (unique in case $\mu(\Lambda) = n + 1 - |\beta|$) satisfying the conditions (1.11). It is clear that the polynomial

$$P^*_{\Lambda,\beta} := \tilde{P}_{\Lambda,\beta} + \bar{P}_{\Lambda,\beta} \in \Pi_{n-|\beta|}(\Lambda)$$

has the following properties:

$$D^j_{d_\Lambda} P^*_{\Lambda,\beta}(\lambda) = P^\beta_{\lambda,j} \qquad \text{for} \quad (\lambda, j) \in \operatorname{dom} \mathscr{D}^{k,\beta}(\Lambda) \text{ and proper } \lambda, \tag{1.13}$$

and

$$(P^*_{\Lambda,\beta})^{[n-j]} = P^\beta_{\Lambda_\infty,j} \qquad \text{for} \quad j = 0, ..., \mu_\infty - 1. \tag{1.14}$$

*Remark* 1.3.1. The necessity of condition (d) for interpolation problem (1.1) to be solvable is evident. The choice in this case is

$$\tilde{P}_{\Lambda,\beta} = D^\beta_{\Lambda^\perp} P|_\Lambda - P_{\Lambda_\infty,\beta}.$$

We handle the definition of consistency for an *improper* $\Lambda$ by reducing it to the proper case as follows. Let $2 \leqslant s \leqslant k$ and let $\Lambda \in \mathscr{L}^{s-1}$ be improper. Set

$$\mathscr{H}^0 := \{H^0 : H \in \mathscr{H}\}$$

and note that there are no improper manifolds in $\mathscr{L}^s_{\mathscr{H}^0}$. Consider the following sequence of polynomials on manifolds in $\mathscr{L}^s_{\mathscr{H}^0}$:

$$\mathscr{D}^{s,\beta}_{\mathscr{H}^0}(\Lambda^0) := (P_{\lambda^0,j} := P_{\lambda,j} \colon (\lambda, j) \in \operatorname{dom} \mathscr{D}^{s,\beta}(\Lambda)).$$

DEFINITION 1.4. For an improper $\Lambda \in \mathscr{L}^{s-1}$, we say that the derived sequence $\mathscr{D}^{s,\beta}(\Lambda)$ is **consistent** if the sequence $\mathscr{D}^{s,\beta}_{\mathscr{H}^0}(\Lambda^0)$ is consistent at $\Lambda^0$, which in this case is equivalent to condition (a) only.

DEFINITION 1.5. We say that the data sequence $\mathscr{D}^s$ is **consistent** if, for each $\Lambda \in \mathscr{L}^{s-1}$ and each $\beta \in \mathbb{Z}^{s-1}_+$ with $|\beta| \leqslant \# \mathscr{H}_\Lambda - (s-1)$, the derived sequence $\mathscr{D}^{s,\beta}(\Lambda)$ is consistent at $\Lambda$.

## 2. THE MAIN THEOREM AND THE CASE $s = 1$

THEOREM 2.1. *Let $\mathscr{D}^s = \mathscr{D}^s_{\mathscr{H}}$ be the polynomial sequence introduced earlier, for some $1 \leqslant s \leqslant k$. A necessary and sufficient condition for the existence of a polynomial $P \in \Pi_n(\mathbb{R}^k)$ such that*

$$D^{\alpha}_{\lambda^\perp} P|_\lambda = P_{\lambda,\alpha} \qquad \textit{for all} \quad (\lambda, \alpha) \in \operatorname{dom} \mathscr{D}^s$$

*is the consistency of $\mathscr{D}^s$.*

*The polynomial $P$ is unique iff $\# \mathscr{H} \geqslant n + s$.*

Our next purpose is to complete the proof of Theorem 2.1 in the case $s = 1$, $k > 1$ (the case $k = 1$ reduces to Lagrange–Hermite univariate interpolation).

Let

$$\mathscr{H} =: (H_1, ..., H_\mu) =: \begin{Bmatrix} L_1 \\ \mu_1 \end{Bmatrix}, ..., \begin{Bmatrix} L_r \\ \mu_r \end{Bmatrix},$$

where we assume that $H_1 = L_1, ..., H_\mu = L_r$ (the distinct hyperplanes $L_i$ occur in $\mathscr{H}$ with respective multiplicities $\mu_i$ and have unit normals $n_i$), and let

$$\mathscr{D}(\mathscr{H}) := \mathscr{D}^1_{\mathscr{H}} = (P_{i,j} \colon j = 0, ..., \mu_i - 1,\ i = 1, ..., r).$$

(Here we have $\mu_\infty = 0$.)

Actually, in this case of $s = 1$, we will discuss a *more general* setting, namely, we put

$$\mu_i =: \bar{\mu}_i + m, \qquad i = 1, ..., r,$$

where $m$ is an *arbitrary integer* with $0 \leqslant m < \min_{1 \leqslant i \leqslant r} \mu_i$ and, correspondingly,

$$u := (n + 2 - \bar{\mu} - m)_+ \cdot \text{signum}\, m, \qquad \text{with} \quad \bar{\mu} := \sum_{i=1}^{r} \bar{\mu}_i.$$

THEOREM 2.2. *Let $\mathscr{D}(\mathscr{H})$ be consistent (i.e., conditions* (a), (b) *with $A = \mathbb{R}^k$, $s = 1$, and $\beta = 0$ hold). Then there exists a polynomial $P \in \Pi_n(\mathbb{R}^k)$ such that*

$$D^j_{n_i} P |_{L_i} = P_{i,j}, \qquad \forall (i, j) \in \operatorname{dom} \mathscr{D}(\mathscr{H}). \tag{2.1}$$

*The polynomial $P$ is unique iff $\mu \geqslant n + 1$.*

*Proof.* The uniqueness of the polynomial $P$ in the case $\mu \geqslant n + 1$ follows from the following well-known Lemma 2.3 (see, for example, [3]). Here and below, we denote by $\rho(x, L)$ the signed distance of $x \in \mathbb{R}^k$ from the proper hyperplane $L$ in $\mathbb{R}^k$; in particular, $L = \{x \in \mathbb{R}^k : \rho(x, L) = 0\}$.

LEMMA 2.3. *Let $Q \in \Pi_n$ and*

$$D^j_{L^\perp} Q |_L = 0, \qquad \textit{for} \quad j = 0, ..., v - 1.$$

*Then, for some $q \in \Pi_{n-v}(\mathbb{R}^k)$,*

$$Q = \rho(\cdot, L)^v\, q.$$

On the other hand, if $\mu \leqslant v$, then the polynomial $\prod_{H \in \mathscr{H}} \rho(\cdot, H) \in \Pi_n(\mathbb{R}^k)$ satisfies condition (2.1) with all $P_{i,j} = 0$, which means that the polynomial $P$ satisfying (2.1) is not unique.

Let us now construct the interpolating polynomial $P$ in the case $s = 1$. We first consider the case $\mu \leqslant n + 1$. Here we have

$$u = (n + 2 - \bar{\mu} - m) \cdot \text{signum}\, m,$$

since $\mu = \bar{\mu} + rm \geqslant \bar{\mu} + m$. This in the case $m \geqslant 1$ implies

$$\bar{\mu}(l) + m + u - 2 = \bar{\mu}(l) + n - \bar{\mu} \geqslant \bar{\mu}(l) + \mu - \bar{\mu} - 1 \geqslant \bar{\mu}(l) + rm - 1 \geqslant \mu(l) - 1$$

and therefore $\mathscr{D}(\mathscr{H})$ is fully consistent at $\mathbb{R}^k$.

For proper $L$ in $\mathbb{R}^k$ of dimension $k - 1$ and $q \in \Pi_n(L)$ we denote by $\check{q}$ the polynomial from $\Pi_n$ satisfying the conditions

$$\check{q}|_L = q, \qquad \check{q}(x_1) = \check{q}(x_2) \quad \text{if} \quad x_1 - x_2 \in L^\perp.$$

Let us prove, by induction on $\mu$, that there exist polynomials $q_i \in \Pi_{n-i+1}$, $i=1, ..., \mu$, such that the polynomial (cf. [2])

$$P = P_\mu := q_1 + \rho(\cdot, H_1) q_2 + \cdots + [\rho(\cdot, H_1) \cdot \cdots \cdot \rho(\cdot, H_{\mu-1})] q_\mu$$

satisfies condition (2.1).

For $\mu = 1$ we put $P_1 = q_1 = \check{P}_{1,0}$. Let $q_1, ..., q_{\mu-1}$ be known and $P_{\mu-1}$ satisfy the conditions

$$D^j_{n_i} P_{\mu-1}|_{L_i} = P_{i,j}, \qquad i=1, ..., r; \quad j=0, ..., \mu_i - 1; \quad (i,j) \neq (r, \mu_r - 1). \tag{2.2}$$

It is enough to find a polynomial $q_\mu \in \Pi_{n-\mu+1}(\mathbb{R}^k)$ such that

$$D^{\mu_r-1}_{n_r}[P_{\mu-1} + \rho(\cdot, H_1) \cdot \cdots \cdot \rho(\cdot, H_{\mu-1}) \cdot q_\mu]|_{L_r} = P_{r,\mu_r-1}. \tag{2.3}$$

It is clear that

$$\begin{aligned} &D^{\mu_r-1}_{n_r}[\rho(\cdot, H_1) \cdot \cdots \cdot \rho(\cdot, H_{\mu-1}) \cdot q_\mu]|_{L_r} \\ &\quad = (\mu_r - 1)! \left[ q_\mu \cdot \prod_{\substack{i=1 \\ H_i \neq H_\mu}}^{\mu-1} \rho(\cdot, H_i) \right] \Bigg|_{L_r}, \end{aligned}$$

and therefore (2.3) reduces to

$$(\mu_r - 1)! \left[ q_\mu \cdot \prod_{\substack{i=1 \\ H_i \neq H_\mu}}^{\mu-1} \rho(\cdot, H_i) \right] \Bigg|_{L_r} = P_{r,\mu_r-1} - D^{\mu_r-1}_{n_r} P_{\mu-1}|_{L_r}. \tag{2.4}$$

Let

$$\begin{aligned} I_0 &:= \{i \colon 1 \leqslant i \leqslant \mu,\ H_i^0 = H_\mu^0,\ H_i \neq H_\mu\}, \\ I_1 &:= \{i \colon 1 \leqslant i \leqslant \mu,\ H_i^0 \neq H_\mu^0\}, \\ \delta_i &:= \# I_i, \qquad i = 0, 1. \end{aligned}$$

Now condition (b) of full consistency of $\mathscr{D}(\mathscr{H})$ at $\mathbb{R}^k$, Remark 1.1.2, and the induction hypothesis yield

$$P_{r,\mu_r-1} - D^{\mu_r-1}_{n_r} P_{\mu-1}|_{L_r} \in \Pi_{n-\mu_r-\delta_0+1}(L_r).$$

Indeed, according to (1.9) for $v = 0, ..., \delta_0 + \mu_r - 2$, and Remark 1.1.2, the $\delta_0$ upper homogeneous components of $P_{r,\mu_r-1}$ are uniquely determined by means of quantities on the right-hand side of (2.2). On the other hand, the polynomial

$$D^{\mu_r-1}_{n_r} P_{\mu-1}|_{L_r}$$

has the same $\delta_0$ upper components, because of the necessity of conditions (1.9) (see Remark 1.1.1) and (2.2). Let us denote

$$\{L_i \cap L_\mu : i \in I_1\} =: \left\{\begin{matrix} l_1 \\ v_1 \end{matrix}, ..., \begin{matrix} l_{r'} \\ v_{r'} \end{matrix}\right\}, \qquad \sum_{i=1}^{r'} v_i = \delta_1.$$

According to condition (a) of consistency and (2.2), as in the previous case, we have

$$D^{j}_{l_i^{\perp}(H_\mu)}[P_{r,\mu_r-1} - D^{\mu_r-1}_{n_r} P_{\mu-1}]|_{l_i} = 0, \qquad j = 0, ..., v_i - 1, \quad i = 1, ..., r'.$$

Hence by Lemma 2.3 we get the factorization

$$[P_{r,\mu_r-1} - D^{\mu_r-1}_{n_r} P_{\mu-1}]|_{L_r} = \left[q \prod_{i=1}^{r'} \rho^{v_i}(\cdot, l_i)\right]\Bigg|_{L_r} = \left[q \prod_{i \in I_1} \rho(\cdot, H_i)\right]\Bigg|_{L_r},$$

where $q \in \Pi_{n-\mu+1}(L_r)$ since $\delta_0 + \delta_1 + \mu_r = \mu$.

It is evident that we will have (2.4), taking $q_\mu = (1/c)\check{q}$, where

$$c = (\mu_r - 1)! \left[\prod_{i \in I_0} \rho(\cdot, H_i)\right]\Bigg|_{L_r} = \text{const}.$$

Let us now consider the case $\mu \geqslant n+2$. We start by choosing a subcollection

$$\tilde{\mathscr{H}} = \left\{\begin{matrix} L_1 \\ \sigma_1 \end{matrix}, ..., \begin{matrix} L_r \\ \sigma_r \end{matrix}\right\} \subset \mathscr{H}, \qquad \sum_{i=1}^{r} \sigma_i = n+1,$$

such that

$$\sigma_i \leqslant \bar{\mu}_i, \qquad i = 1, ..., r \text{ if } \bar{\mu} \geqslant n+1 \text{ (case 1)},$$

and

$$\sigma_i \geqslant \bar{\mu}_i, \qquad i = 1, ..., r \text{ if } \bar{\mu} \leqslant n+1 \text{ (case 2)},$$

Let us check that the polynomial sequence $\mathscr{D}(\tilde{\mathscr{H}})$ is fully consistent at $\mathbb{R}^k$. This is obvious in case 1. For case 2 it is enough to note that

$$\sum_{i=1}^{r} (\sigma_i - \bar{\mu}_i) = n + 1 - \bar{\mu} < m + u,$$

since

(i) if $u = 0$, i.e., $n + 2 - \bar{\mu} - m \leqslant 0$, we have $n + 1 - \bar{\mu} \leqslant m - 1$,

(ii) if $u \neq 0$, i.e., $n + 2 - \bar{\mu} - m = u$, we have $n + 1 - \bar{\mu} = m + u - 1$.

Hence $\mathscr{D}(\tilde{\mathscr{H}})$ is fully consistent at $\mathbb{R}^k$ and using the previous construction we can find a polynomial $\tilde{P} \in \Pi_n$ satisfying the conditions

$$D_{n_i}^j \tilde{P}|_{L_i} = P_{i,j}, \qquad j = 0, ..., \sigma_i - 1, \quad i = 1, ..., r.$$

Let us prove that $\tilde{P}$ is the desired polynomial, i.e., conditions (2.1) are satisfied with $P = \tilde{P}$. Let

$$(r, j) \in \operatorname{dom} \mathscr{D}(\mathscr{H}), \qquad j_0 \geqslant \sigma_r.$$

Assume that (induction on $j_0$)

$$D_{n_r}^j P|_{L_r} = P_{r,j}, \qquad j = 0, ..., j_0 - 1$$

Let us define for $\tilde{\mathscr{H}}$ similar to the case of $\mathscr{H}$:

$$\tilde{I}_0 := I_0 \cap \{i: H_i \in \tilde{\mathscr{H}}\}, \qquad \tilde{\delta}_0 := \# \tilde{I}_0,$$
$$\tilde{I}_1 := I_1 \cap \{i: H_i \in \tilde{\mathscr{H}}\}, \qquad \tilde{\delta}_1 := \# \tilde{I}_1.$$

Then by condition (b) of consistency of $\mathscr{D}(\mathscr{H})$, we have that

$$P_{r,j_0} - D_{n_r}^{j_0} P \in \Pi_{n - j_0 - \eta}(L_r),$$

where

$$\eta = \sum_{i \in \tilde{I}_0} \sigma_i, \qquad \text{or} \qquad \eta = \sum_{i \in \tilde{I}_0} \bar{\mu}_i + (\bar{\mu}_r + m + u - j_0 - 1). \tag{2.5}$$

Exactly we have

$$\eta = \begin{cases} \sum_{i \in \tilde{I}_0} \sigma_i, & \text{in case 1}, \\ \sum_{i \in \tilde{I}_0} \bar{\mu}_i + \min \left\{ \sum_{i \in \tilde{I}_0} (\sigma_i - \bar{\mu}_i),\ \bar{\mu}_r + m + u - j_0 - 1 \right\} & \text{in case 2}. \end{cases}$$

This follows from the fact that in the case 1 the subcollection

$$\left\{ \frac{L_1}{\sigma_1}, ..., \frac{L_{r-1}}{\sigma_{r-1}}, \frac{L_r}{j_0 + 1} \right\}$$

is fully consistent, since

$$j_0 + 1 - \bar{\mu}_r \leqslant \mu_r - \bar{\mu}_r = m,$$

and in case 2 above, the subcollection is fully consistent iff

$$\sum_{i \in \tilde{I}_0} (\sigma_i - \bar{\mu}_i) + j_0 + 1 - \bar{\mu}_r \leqslant m + u.$$

Let us denote now

$$\{H_i \cap H_\mu : i \in \tilde{I}_1\} =: \left\{ \begin{matrix} \tilde{l}_1 \\ \zeta_1 \end{matrix}, ..., \begin{matrix} \tilde{l}_{r_0} \\ \zeta_{r_0} \end{matrix} \right\}, \qquad \sum_{i=1}^{r_0} \zeta_i = \tilde{\delta}_1,$$

and $\tilde{n}_i = \tilde{l}_i^{\perp}(H_\mu)$, $i = 1, ..., r_0$. Then (2.5) and the consistency of $\mathscr{D}(\mathscr{H})$ imply

$$D_{\tilde{n}_i}^j [P_{r_{j_0}} - D_{n_r}^{j_0} \tilde{P}] |_{\tilde{l}_i} = 0, \qquad j = 0, ..., \xi_i - 1, \quad i = 1, ..., r_0,$$

where for $\xi := \sum_{i=1}^{r_0} \xi_i$ we have

$$\xi = \sum_{i \in \tilde{I}_1} \sigma_i, \qquad \text{or} \qquad \xi \geqslant \sum_{i \in \tilde{I}_1} \bar{\mu}_i + (\bar{\mu}_r + m + u - j_0 - 1).$$

Now it is not hard to check that

$$\eta + \xi = \sum_{i \in \tilde{I}_0} \sigma_i + \sum_{i \in \tilde{I}_1} \sigma_i = n + 1 - \sigma_r$$

or

$$\eta + \xi = \sum_{i \in \tilde{I}_0} \bar{\mu}_i + \sum_{i \in \tilde{I}_1} \bar{\mu}_i + (\bar{\mu}_r + m + u - j_0 - 1) = \bar{\mu} + m + u - j_0 - 1.$$

This combined with (i) and (ii) implies

$$\eta + \xi \geqslant n + 1 - j_0, \qquad \text{i.e.,} \quad \xi \geqslant n - j_0 - \eta + 1.$$

The latter in its turn yields

$$P_{r, j_0} = D_{n_r}^{j_0} \tilde{P}$$

Thus the proof of the Theorem 2.2 is complete. ∎

Repeating the proof of Theorem 2.2 for a particular case we get

COROLLARY 2.4. *Let the hyperplanes in $\mathscr{H}$ all be proper and contain the origin, i.e., $\mathscr{H} = \mathscr{H}^0$, and polynomials of $\mathscr{D}(\mathscr{H})$ all be homogeneous:*

$$P_{i,j} \in \Pi_{n-j}^{\infty}(\mathbb{R}^k), \qquad (i, j) \in \operatorname{dom} \mathscr{D}(\mathscr{H}).$$

*Then there exists $P \in \Pi_n^{\infty}(\mathbb{R}^k)$ satisfying condition* (2.1) *if and only if $\mathscr{D}(\mathscr{H})$ is consistent (which in this situation reduces to condition* (a) *for $s = 1$, $\Lambda = \mathbb{R}^k$, and $\beta = 0$).*

*The polynomial $P$ is unique iff $\mu \geqslant n + 1$.*

## 3. The Proof of the Main Theorem

We will prove the sufficiency part of Theorem 2.1 by induction on $n+k$. In the case $\mu < n+s$ we start by adding a hyperplane $L$ to the collection $\mathscr{H}$ and polynomials on manifolds in $\mathscr{L}^s(L, \mathscr{H} \cup \{L\})$ to the sequence $\mathscr{D}^s_{\mathscr{H}}$, such that the resulting polynomial sequence is still consistent. We choose the additional hyperplane $L$ such that

(i) $L$ does not contain any point from $\mathscr{L}^k$,

(ii) $\mathscr{L}^{k-1} \cap L$ consists only of proper points.

In the case $s=k$ we define values and corresponding derivatives of desired polynomials on (proper) points of $\mathscr{L}^s(L, \mathscr{H} \cup \{L\})$ arbitrarily. Now let

$$\lambda' \in \mathscr{L}^s(L, \mathscr{H} \cup \{L\}) \backslash \mathscr{L}^s, \quad s \leqslant k-1, \quad \text{and} \quad l \in \mathscr{L}^{s+1}(\lambda', \mathscr{H} \cup \{L\}).$$

The above assumption then implies that $l = \lambda \cap \lambda'$, with $\lambda \in \mathscr{L}^s$. We first define polynomials on $l$ as

$$P^i_l := \left\{ \sum_{j+|\alpha|=i} c^i_{j,\alpha} D^j_{l^\perp(\lambda)} P_{\lambda,\alpha} \right\} \Bigg|_l, \qquad i \leqslant \# \mathscr{L}_\lambda - s, \tag{3.1}$$

where coefficients $c^i_{j,\alpha}$ are found from the relation

$$D^i_{l^\perp(\lambda')} = \sum_{j+|\alpha|=i} c^i_{j,\alpha} D^j_{l^\perp(\lambda)} D^\alpha_{\lambda^\perp}.$$

In the case $s \leqslant k-2$ the consistency on $\Lambda$ of polynomials just defined clearly follows from the consistency of $\mathscr{D}^s_{\mathscr{H}}$, while for $s=k-1$ consistency condition (d) coincides with univariate Hermite interpolation by polynomials of degree $\leqslant n$ with $\mu + 1 - (k-1) \leqslant n$ parameters. Hence by induction we can define a polynomial $P_{\lambda',0} \in \Pi_n(\lambda')$ such that

$$D^i_{l^\perp(\lambda')} P_{\lambda',0}|_l = P^i_l \qquad \text{for all} \quad l \in \mathscr{L}^{s+1}(\lambda', \mathscr{H} \cup \{L\}).$$

In its turn, this condition ensures (since (3.1) holds and $\#\mathscr{L}_l - s - 1 = \#\mathscr{L}_\lambda - s$ for the above $l, \lambda$) full consistency along $l = \lambda \cap \Lambda$ and therefore the consistency in general of the resulting sequence, $\mathscr{D}^s_{\mathscr{H} \cup \{L\}}$.

*Hence we can, without loss of generality, restrict ourselves to the case* $\mu \geqslant n+s$.

Now, on account of Theorem 2.2, to finish the proof it is sufficient to construct a polynomial sequence $\mathscr{D}^{s-1} = \mathscr{D}^{s-1}_{\mathscr{H}}$ such that for each $P \in \Pi_n(\mathbb{R}^k)$ the following conditions are equivalent:

(1) $D^\alpha_{\lambda^\perp} P|_\lambda = P_{\lambda,\alpha}$ for all $(\lambda, \alpha) \in \operatorname{dom} \mathscr{D}^s$,

(2) $D^\beta_{\Lambda^\perp} P|_\Lambda = P_{\Lambda,\beta}$ for all $(\Lambda, \beta) \in \operatorname{dom} \mathscr{D}^{s-1}$.

We start the construction with the case $1 \leqslant s \leqslant k-1$. Let us define $P_{\Lambda,\beta}$ for $(\Lambda, \beta) \in \operatorname{dom} \mathscr{D}^{s-1}$ with proper $\Lambda$. If $\Lambda_\infty \notin \mathscr{L}^s$, then we find $P_{\Lambda,\beta} \in \Pi_{n-|\beta|}(\Lambda)$ according to Theorem 2.2, by the conditions

$$D^j_{\lambda^\perp(\Lambda)} P_{\Lambda,\beta}|_\lambda = P^\beta_{\lambda,\Lambda,j} \qquad \text{for all} \quad (\lambda, j) \in \operatorname{dom} \mathscr{D}^{s,\beta}(\Lambda). \tag{3.2}$$

In the case $\Lambda_\infty \in \mathscr{L}^s$, relation (1.6), on account of (1.5), uniquely determines the $\# \mathscr{H}_{\Lambda_\infty} - s - |\beta| + 1$ highest homogeneous components of $P_{\Lambda,\beta}$, for which we denote the sum by $\bar{P}_{\Lambda,\beta}$. After this, we define the second polynomial $\tilde{P}_{\Lambda,\beta} \in \Pi_{n - \# \mathscr{H}_{\Lambda_\infty} + s - 1}$ according to Theorem 2.2, by the conditions

$$D^j_{\lambda^\perp(\Lambda)} \tilde{P}_{\Lambda,\beta} = P^\beta_{\lambda,j} - \bar{P}^\beta_{\Lambda,\beta}|_\lambda, \qquad (\lambda, j) \in \operatorname{dom} \mathscr{D}^{s,\beta}(\Lambda). \tag{3.3}$$

Relations (3.2) and (3.3) determine $P_{\Lambda,\beta}$ and $\tilde{P}_{\Lambda,\beta}$ uniquely, since in the first case we have

$$\begin{aligned} \# \mathscr{L}^s(\Lambda) &\geqslant \# \mathscr{H} - \# \mathscr{H}_\Lambda + [\# \mathscr{H}_\Lambda - s + 1 - |\beta|] \\ &= \# \mathscr{H} - s - |\beta| + 1 \geqslant n - |\beta| + 1, \end{aligned}$$

and in the second case

$$\# \mathscr{L}^s(\Lambda) \geqslant \# \mathscr{H} - \# \mathscr{H}_{\Lambda_\infty} + [\# \mathscr{H}_\Lambda - s + 1 - |\beta|] \geqslant n - \# \mathscr{H}_{\Lambda_\infty} + s.$$

For the polynomial $P_{\Lambda,\beta} := \bar{P}_{\Lambda,\beta} + \tilde{P}_{\Lambda,\beta}$ we have

$$D^j_{\lambda^\perp(\Lambda)} P_{\Lambda,\beta}|_\lambda = P^\beta_{\lambda,j} \qquad \text{for all} \quad (\lambda, j) \in \operatorname{dom} \mathscr{D}^{s,\beta}(\Lambda). \tag{3.4}$$

and

$$P^{[n-j]}_{\Lambda,\beta} = P^\beta_{\Lambda_\infty,j}, \qquad j \leqslant \# \mathscr{H}_{\Lambda_\infty} - s - |\beta|. \tag{3.5}$$

In the case of improper $\Lambda \in \mathscr{L}^{s-1}$, $2 \leqslant s \leqslant k$, $|\beta| \leqslant \# \mathscr{H}_\lambda - s + 1$ (then all $\lambda \in \mathscr{L}^s(\Lambda)$ are improper too) conditions (3.2) uniquely determine a homogeneous polynomial of degree $n - |\beta|$ on $\Lambda^0$ according to Corollary 2.4 since

$$\begin{aligned} \# \mathscr{L}^s(\Lambda) &\geqslant \# \mathscr{H} - \# \mathscr{H}_\Lambda + [\# \mathscr{H}_\Lambda - s + 1 - |\beta|] \\ &= \# \mathscr{H} - s - |\beta| + 1 \geqslant n - |\beta| + 1. \end{aligned}$$

In the case $s = k$, in addition to the above-mentioned polynomials on improper hyperplanes, the sequence $\mathscr{D}^{s-1}$ consists of additional polynomials on proper hyperplanes:

$$P_{\Lambda,\beta} := P^*_{\Lambda,\beta}, \qquad \text{for} \quad (\Lambda, \beta) \in \operatorname{dom} \mathscr{D}^{s-1} \text{ with } \Lambda \text{ proper}.$$

Now let us check the consistency conditions for $\mathscr{D}^{s-1}$. Let $V \in \mathscr{L}^{s-2}$ be proper. Then, for $\lambda \in \mathscr{L}^s(V)$, we have order of consistency

$$\bar{\mu}_\lambda(V) + m_V + u_V - 2 = \# \mathscr{H}_\lambda - s - |\beta|, \tag{3.6}$$

since $u_V = 0$. To check condition (a), in view of (1.3), (3.2), and (3.6), we take

$$Q^\beta_{\lambda,\nu}(t) = \sum_{|\alpha| = \nu} c_\alpha P_{\lambda,\alpha}, \qquad t \in \lambda^\perp(V),$$

where the coefficients $c_\alpha = c_\alpha(t)$ are found from the relation

$$D^\beta_{V\perp} D^\nu_t = \sum_{|\alpha| = \nu + |\beta|} c_\alpha D^\alpha_{l^\perp(V)}.$$

Condition (b), in view of (3.6), is satisfied with

$$Q^\beta_{\lambda,\nu}(t) = P_\nu^{\lambda^\perp(V)}|_V, \qquad \text{for improper} \quad \lambda \in \mathscr{L}^s(V),$$

where

$$P_\nu = \bar{P}^\beta_{\lambda_\infty,\nu}.$$

Here, we use relation (3.5) in the case $1 \leqslant s \leqslant k-1$, and (1.14) in the case $s = k$. Condition (c) follows from relation (3.2) and consistency condition (b) of $\mathscr{D}^s$ if $1 \leqslant s \leqslant k-1$, and from relations (1.12), (1.14) if $s = k$.

In the case of improper $V \in \mathscr{L}^{s-2}$, we need only check condition (a) for $\mathscr{D}^s(V^0)$, which can be done similarly to the previous case by using relation (3.6). Now the implication (1) $\Rightarrow$ (2) readily follows from the uniqueness of $\mathscr{D}^{s-1}$, while the opposite implication is ensured by way of construction of $\mathscr{D}^{s-1}$. Indeed, to determine $P_{\lambda,\alpha}$ for some $(\lambda, \alpha) \in \operatorname{dom} \mathscr{D}^s$ by means of polynomials from $\mathscr{D}^{s-1}$, we consider the collection of $(s-1)$-dimensional manifolds $\mathscr{H}_\lambda \cap \lambda^\perp$ in the space $\lambda^\perp$, $\dim \lambda^\perp = s$ and the interpolation of degree $i \leqslant \mu(l) - s$ to traces $(D^i_t P|_\lambda : t \in \lambda^\perp)$ obtained with the help of the polynomials induced on lines $\{\mathscr{H}_\lambda \cap \lambda^\perp\}^{s-1}$.

Obviously the uniqueness condition in Theorem 2.1 is satisfied here, since it turns into the following inequality:

$$\mu(\lambda) \geqslant i + (s-1).$$

Hence, $P_{\lambda,\alpha}$ is determined uniquely. This finishes the proof of Theorem 2.1. ∎

## 4. Interpolation on the Sphere by Homogeneous Polynomials

The analog on the sphere of interpolation considered will be obtained by applying Theorem 2.1 to a collection $\mathscr{H}$ with $\mathscr{H} = \mathscr{H}^0$ and the sequence $\mathscr{D}^s$ consisting of homogeneous polynomials. This case is much simpler, since here we deal only with proper objects.

Let $S$ be a sphere in $\mathbb{R}^k$ centered at the origin. The spherical manifold of dimension $v$, $0 \leqslant v \leqslant k-1$, is defined as the intersection

$$\tilde{H} := H \cap S, \qquad \text{with } \{0\} \in H, \quad \dim H = v + 1.$$

Let

$$\tilde{\mathscr{H}} = \{\tilde{H} : H \in \mathscr{H}\}.$$

By $\tilde{\mathscr{L}}^s$, $0 \leqslant s \leqslant k-1$, we denote the set of all $(k-1-s)$-dimensional (spherical) manifolds which are intersections with manifolds from $\tilde{\mathscr{H}}$. Define

$$\Pi_n(\tilde{H}) := \{P|_{\tilde{H}} : P \in \Pi_n^\infty(H)\}.$$

For $f$ defined on $S$, and $y$ a tangential direction to $S$, let

$$D_y f := \lim_{t \to 0} \frac{f(\cdot + t\tilde{y}) - f}{t},$$

where $(\cdot + t\tilde{y})$ is the intersection of $S$ and the line between $(\cdot + ty)$ and the origin. For manifold $\tilde{I} \subset \tilde{H}$ and multi-index $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}_+^m$, $m = \dim \tilde{H} - \dim \tilde{I}$,

$$D^\alpha_{I^\perp(\Lambda)} f := D^{\alpha_1}_{I_1^\perp(\Lambda)} \cdots D^{\alpha_m}_{I_m^\perp(\Lambda)} f.$$

Suppose we have the polynomial sequence

$$\tilde{\mathscr{D}}^s := (P_{\tilde{\lambda},\alpha} \in \Pi_{n-|\alpha|}(\tilde{\lambda}) : \tilde{\lambda} \in \tilde{\mathscr{L}}^s, \; \alpha \in \mathbb{Z}_+^s, \; |\alpha| \leqslant \# \mathscr{H}_{\tilde{\lambda}} - s).$$

Notations and definitions used in this part without special mention are similar to previous ones despite the simplicity due to the absence of improper cases.

Definition 4.1. The sequence $\tilde{\mathscr{D}}^s$ is said to be consistent if the sequence $\mathscr{D}^{s,\beta}(\tilde{\Lambda})$ is consistent at $\tilde{\Lambda}$ for every

$$\tilde{\Lambda} \in \tilde{\mathscr{L}}^{s-1}, \qquad \beta \in \mathbb{Z}_+^{s-1}, \qquad |\beta| \leqslant \# \mathscr{H}_{\tilde{\lambda}} - s + 1.$$

The latter in the cases $0 \leqslant s \leqslant k-2$ and $s = k-1$ means that the following conditions (a') and (d') hold respectively:

(a′) for every $\tilde{l} \in \mathscr{L}^{s+1}(\tilde{\Lambda})$ there exist polynomials

$$Q^{\beta}_{\tilde{l},\nu} \in \Pi^{\infty}_{\nu}(l^{\perp}(\Lambda), \tilde{l}), \qquad \nu = 0, ..., v_l(\Lambda)$$

satisfying the relation

$$D^{j}_{n_i} Q^{\beta}_{\tilde{l},\nu}(t) = \frac{\nu!}{(\nu - j)!} D^{\nu-j}_{t} P_{i,j}|_{\tilde{l}}$$

for $\lambda \in \mathscr{L}^{s}_{l}(\Lambda)$, $t \in \tilde{\lambda} \cap l^{\perp}(\Lambda)$, $j = 0, ..., \min\{\nu, \mu_\lambda - 1\}$.

(d′) if $\tilde{\Lambda} \in \tilde{\mathscr{L}}^{k-2}$ and $\mu_{\tilde{\Lambda}} \geqslant n + 1 - |\beta|$, then there exists a polynomial $P_{\tilde{\Lambda},\beta} \in \Pi_{n-|\beta|}(\tilde{\Lambda})$ satisfying for $j = 0, ..., (\mu_\lambda - 1 - |\beta|)_+$, $\lambda \in \mathscr{L}^{k-1}(\Lambda)$ the following relation:

$$D^{j}_{d_{\tilde{\lambda}}} P_{\tilde{\Lambda},\beta}(\lambda) = P_{\lambda,j}.$$

THEOREM 4.2. *Let $\tilde{\mathscr{D}}^s$, for some $0 \leqslant s \leqslant k-1$, be the given polynomial sequence. The necessary and sufficient condition for the existence of a polynomial $P \in \Pi_n(S)$ satisfying*

$$D^{\alpha}_{\tilde{\lambda}^{\perp}} P|_{\tilde{\lambda}} = P_{\tilde{\lambda},\alpha}, \qquad \text{for all} \quad (\tilde{\lambda}, \alpha) \in \operatorname{dom} \tilde{\mathscr{D}}^s,$$

*is the consistency of the sequence $\tilde{\mathscr{D}}^s$.*

*The polynomial $P$ is unique iff $\mu \geqslant n + s$.*

## 5. Special Cases and Consequences

From Theorem 2.1 (with $s = k$) it is not hard to obtain the following two pointwise interpolations ($s = k$), which do not involve any consistency conditions.

Let $\mathscr{H}$ be a collection of hyperplanes with $\#\mathscr{H} = n + k$ and assume that $\mathscr{L}^k$ consists of only proper points; i.e., every $k$ hyperplanes from $\mathscr{H}$ have exactly one common (proper) point.

### 5.1. *Chung–Yao Interpolation (See [1])*

Assume that the multiplicity of every point of $\mathscr{L}^k$ equals $k$, i.e., $\#\mathscr{L}^k = \binom{n+k}{k}$.

*Then for an arbitrary sequence of values*

$$\mathscr{D}^k := \{c_\lambda \in \mathbb{R} \colon \lambda \in \mathscr{L}^k\},$$

*there exists a unique polynomial $P \in \Pi_n$ such that*

$$P(\lambda) = c_\lambda \qquad \text{for all} \quad \lambda \in \mathscr{L}^k.$$

5.2. *Hakopian Interpolation* (*See* [3])

This interpolation, actually, is the generalization of the previous one to the Hermite case:

*For an arbitrary real number sequence*

$$\mathscr{D}^k := \{c_\lambda^\alpha \in \mathbb{R} : |\alpha| \leqslant \#\mathscr{H}_\lambda - k,\ \lambda \in \mathscr{L}^k\},$$

*there exists a unique polynomial* $P \in \Pi_n$ *such that*

$$D^\alpha P(\lambda) = c_\lambda^\alpha \qquad \text{for all} \quad (\lambda, \alpha) \in \operatorname{dom} \mathscr{D}^k.$$

Of course, on account of Theorem 2.1, one can easily omit the common restriction of these two interpolations to $\mathscr{L}^k$ with only proper points and consider the case of improper interpolations, too.

5.3. *Tensor-Product Interpolation*

Now we choose the collection of distinct hyperplanes

$$\mathscr{H} = \{H_{i,j} : j = 0, ..., n_i,\ i = 1, ..., k\}, \qquad \sum_{i=1}^{k} n_i = n,$$

where $H_{i,j}$ is given by the equation $x_i = a_{i,j}$.

The only improper points of $\mathscr{L}^k$ are $l^i_\infty$, where $l^i$ is $i$th axis of $\mathbb{R}^k$, and $\#\mathscr{H}_{l^i_\infty} = n - n_i$. It is not hard to verify that the interpolation conditions of $P \in \Pi_n$ at these improper points fix the coefficients of monomials from $\Pi_n \backslash \Pi_{\bar{n}}$, where

$$\Pi_{\bar{n}} := \left\{ \sum_{\alpha \leqslant \bar{n}} c_\alpha x^\alpha : c_\alpha \in \mathbb{R} \right\}, \qquad \bar{n} := (n_1, ..., n_k).$$

Therefore, taking these conditions as equal to zero, we obtain:

*For an arbitrary sequence of real numbers*

$$T = (c_\alpha \in \mathbb{R} : \alpha \in \mathbb{Z}^k_+,\ \alpha \leqslant \bar{n})$$

*there exists a unique polynomial* $P \in \Pi_{\bar{n}}$ *such that*

$$P(a_{1,\alpha_1}, ..., a_{k,\alpha_k}) = c_\alpha \qquad \text{for all} \quad \alpha \in \operatorname{dom} T.$$

Using Theorem 2.1, one can consider mixtures of such interpolations, too.

For example, if $n_1 + 1$ hyperplanes of $\mathscr{L}$ coincide with the above $H_{1,j}$, $j = 0, ..., n_1$, and the collection $\mathscr{H} \backslash \{H_{1,j}\}_{j=1}^{n_1}$ satisfies the conditions of

Chung–Yao interpolation, then we get correct interpolation from the polynomial space

$$\left\{\sum_{|\alpha| \leqslant n,\ |\alpha| - \alpha_1 \leqslant n - n_1} c_\alpha x^\alpha : c_\alpha \in \mathbb{R}\right\}$$

at the proper points of $\mathscr{L}^k$.

5.4. *Finite Element (F-E) Interpolations*

Here, we will discuss the cases $k = 2$ (in detail) and $k = 3$ of F-E interpolation, i.e., interpolations on a triangle $\mathfrak{T}$ and a tetrahedron $\mathfrak{P}$.

In $(n, v)$ F-E interpolation ($n$ is degree, $v$ is smoothness), the following parameters are given in a way to ensure a $C^v$-matching (belonging to the space $C^v$) of the interpolant polynomials along the common side of adjacent elements in a triangulation:

(i) values of the polynomial $P \in \Pi_n$ and its derivatives (up to order $n_0$) at the vertices of $\mathfrak{T}$ ($\mathfrak{P}$);

(ii) values of the polynomial and its normal derivatives to the sides of $\mathfrak{T}$ (faces and sides of $\mathfrak{P}$);

(iii) values of the polynomial and its derivatives (up to order $c$) at an interior point (center) of $\mathfrak{T}$ ($\mathfrak{P}$).

Let us start with the case $k = 2$. The above setting requires consideration of interpolation to traces of the polynomial $P \in \Pi_n$ and its normal derivatives up to order $v$ on each side of the triangle; i.e., $\mathscr{H}$ consists of three lines, $L_0$, $L_1$, and $L_2$, each of them of multiplicity $v + 1$, and $s = 1$.

In this case, the consistency conditions are reduced to (a) at the vertices of $\mathfrak{T}$. Since the interpolating parameters are arbitrary, we must have full consistency up the derivative order

$$n_0 \geqslant 2v \tag{5.1}$$

at each vertex of triangle. Again, since the values of the parameters are arbitrary, the above consistency will be guaranteed iff the collection of interpolation parameters includes values of $P$ and its $n_0$ derivatives at vertices of $\mathfrak{P}$. To complete the sequence of parameters, one must supply in addition the values of $P$ and its normal derivatives at the sides of the triangle until the unique determination of all traces participating in the above interpolation.

It is obvious that we have the following necessary condition, too:

$$2n_0 + 1 \leqslant n. \tag{5.2}$$

Therefore, (5.1) and (5.2) imply the following necessary condition for the regularity of $(n, \nu)$ F-E interpolation (see [6]):

$$n \geqslant 4\nu + 1. \tag{5.3}$$

Moreover, we obtain the general construction of F-E interpolations in the case $k = 2$ (cf. [5]). Let

$$n = 3(\nu + 1) + c; \qquad c \geqslant -1, \quad \nu \geqslant 0. \tag{5.4}$$

We put

$$r := n + 1 - 2(n_0 + 1),$$

where $n_0$ satisfies conditions (5.1) and (5.2).

Then we take the parameters of (i) and (iii) and normal derivatives of order $i$ at $i + r$ points of each side of the triangle, for $i = 0, ..., \nu$. One can easily verify, using Lemma 2.3 and relation (5.4), that the resulting interpolation is correctly defined if the number of prescribed parameters is equal to $\dim \Pi_n(\mathbb{R}^2)$ and that the latter holds iff

$$n_0 = 2\nu \qquad \text{or} \qquad n_0 = 2\nu + 1.$$

Note that the interpolating polynomial $P \in \Pi_n$ can be found by the formula

$$P = P_c[\rho(\cdot, L_0)\, \rho(\cdot, L_1)\, \rho(\cdot, L_2)]^{\nu+1} + P_{\mathfrak{T}},$$

where $P_{\mathfrak{T}}$ is determined by the parameters on the sides of $\mathfrak{T}$ according to Theorem 2.2, and $P_c \in \Pi_c(\mathbb{R}^2)$ is determined by Taylor interpolation using the parameters at the center:

*Thus we obtain that condition* (5.3) *is necessary and sufficient for the regularity of* $(n, \nu)$ F-E *interpolation on* $\mathfrak{T}$.

Some examples are given in Fig. 1, where we denote the values and derivatives by points and circles, respectively, and (multiple) normal derivatives by (multiple) arrows.

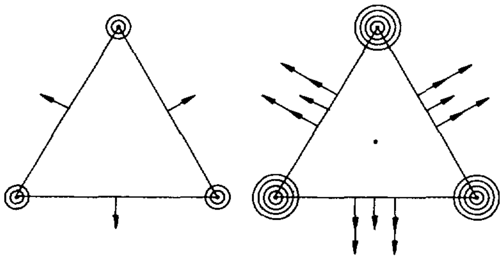
In the case $k = 3$, we consider the analogous interpolation to traces on faces of $\mathfrak{P}$; hence, again $s = 1$. The analog of (5.1) here is

$$n_1 \geqslant 2\nu, \tag{5.5}$$

where $n_1$ is the derivative order of full consistency at each side of the tetrahedron, obtained from condition (a).

We must look for a plane $(n, n_1)$ F-E interpolation; therefore, condition (5.3), namely that $n \geqslant 4n_1 + 1$, combined with (5.5) implies that the condition

$$n \geqslant 8\nu + 1$$

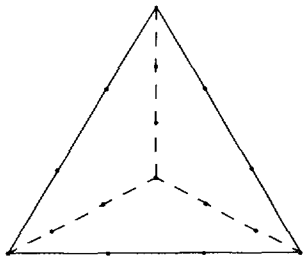


$n=5,\ \nu=1,$
$n_0=2,\ c=-1$
(the Argyris triangle)

$n=9,\ \nu=2,$
$n_0=4,\ c=0$

FIGURE 1

$n=3,\ \nu=0,$
$n_0=0,\ c=-1$



$n=9,\ \nu=1,$
$n_0=4,\ n_1=2,$
$c=1$

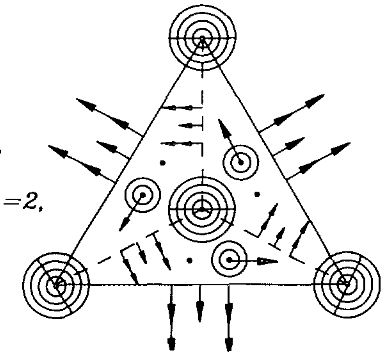


FIGURE 2

is necessary for the existence of 3-space $(n, \nu)$ F-E interpolation. Condition (5.4) in this case is replaced by

$$n = 4(\nu + 1) + c; \qquad c \geq -1, \quad \nu \geq 0$$

and the construction of F-E interpolations is carried out as in the plane. Some examples are given in Fig. 2. Here, the parameters in the frontal side and at the center are omitted. Circles (with straight lines) mean derivatives in faces (in the space) and arrows mean normal derivatives to sides and faces of $\mathfrak{P}$.

## ACKNOWLEDGMENT

We are grateful to Professor C. de Boor for the interest he has shown in our work and his substantial help in improving the presentation of this paper.

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